Partiële differentiaalvergelijkingen, WIPDV-07 2010/11 semester II b

Examination, June 30th, 2011.

Name	**Student number**

Notes:

- You may use one sheet (single side written) with notes from the lectures.
- During the exam it is NOT permitted to consult books, handouts, other notes.
- Numerical/graphic calculators are permitted, symbolic calculators are NOT permited.
- Devices with wireless internet connection and/or document readers are NOT permitted.
- To pass the exam, You need to gather at least half of the total points at the final exam.
- Hint: please describe the solution procedures in full details, not only the results.

TEST (to be returned by 12:00)

1. Determine if the following second-order equations are hyperbolic, elliptic, parabolic.
   (a) **[pts 3]** $u_{xx} - 2u_{xy} + u_{yy} = 0$
   (b) **[pts 3]** $3u_{xx} + u_{xy} + u_{yy} = 0$
   (c) **[pts 4]** $au_{xx} + au_{xy} + u_{yy} = 0$, a integer

2. Consider the initial value problem

$$xu_x + 2yu_y + u_t = 3u, \qquad u(x, y, 0) = f(x, y)$$

   (a) **[pts 2]** Write the characteristic equations for this problem.
   (b) **[pts 5]** Find all solutions of the initial value problem using the method of characteristics.
   (c) **[pts 3]** Consider the special case where the function f is defined as follow

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(v, w) = v + w.$$

   Write the particular solution of the initial value problem for this special case and verify that the solution satisfies the initial value problem.

3. [pts 8] Find all separable solutions $u = F(x)G(y)H(z)$ of the equation $u_x - u_y + u_z = 0$. The boundary conditions are not given.

4. (a) [pts 2] Write the Fourier Sine Series of $f(x) = x^3 - x$ on $[0, 1]$.
   (b) [pts 5] Show that the nth coefficient of the Fourier Sine Series at point (a) has the expression
   $$\frac{12}{n^3\pi^3}(-1)^n.$$
   (c) [pts 3] Evaluate the series found in (a)-(b) at $x = 1/2$ and show that you obtain
   $$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots.$$
   (d) [pts 5] Apply Parseval's identity to your answer in (a)-(b) and show that
   $$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

5. (a) [pts 6] Solve the wave equation $u_{tt} = c^2 u_{xx}$ on the whole real line in the case that the initial position $\phi(x)$ of the string is $\phi(x) = x^2$, and the initial velocity $\psi(x)$ is $\psi(x) = x + 1$.
   (b) [pts 4] Draw graphically the domain of dependence of the solution u at the point $(x_0, t_0) = (2, 1)$ in the case $c = 1$.

6. (a) [pts 2] Recall the definition of harmonic function.
   (b) [pts 3] Verify that $u(x, y) = e^y \cos x$ is harmonic for all (x, y).
   (c) [pts 5] Use the mean value property to show that
   $$\frac{1}{2\pi}\int_0^{2\pi} e^{\sin t}\cos(\cos t))dt = 1.$$
   (d) [pts 5] Find the maximum and minimum values of $u(x, y) = e^y \cos x$ for (x, y) in the rectangular region where $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ and $-1 \leq y \leq 1$.

7. (a) [pts 3] Recall the two Green's identities.
   (b) [pts 5] Suppose that $A \subset \mathbb{R}^n$ is a bounded region and that $u : \mathbb{R}^n \to \mathbb{R}$ satisfies
   $$\Delta u(x) = \lambda u(x), \text{ when } x \in A, \quad \text{and} \quad u(x) = 0 \text{ when } x \in \partial A \text{ (boundary of A)}.$$
   Using the Green's identities, show that u must be identically zero if $\lambda \geq 0$.